Locally moving groups and the reconstruction of structures from their automorphism group

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The plan of this lecture

(1) Reconstruction problems (Explanation).

(2) Statement of a reconstruction theorem for locally compact spaces.

(3) Example: Symmetric groups.

(4) Locally moving groups, definition and examples.

(5) Stetement of the Reconstruction Theorem for Locally Moving Groups.

(6) Proof of a reconstruction theorem for locally compact spaces.

(7) An open question on the reconstruction of locally convex spaces.

(8) Theorems which are consequences of the Reconstruction Theorem for Locally Moving Groups.

Reconstruction problems

Let M and N be mathematical structures of the same type and $\tau : M \cong N$.

Using τ, we define an isomorphism ϕ between $\mathrm{Aut}(M)$ and $\mathrm{Aut}(N)$:

$$\phi(g) = \tau \circ g \circ \tau^{-1} \text{ for all } g \in \mathrm{Aut}(M).$$

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- We ask whether the converse is true. That is: $(\star)$ If $\phi : \mathrm{Aut}(M) \cong \mathrm{Aut}(N)$, does there exist and isomorphism τ between M and N such that

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•• In general the answer is of course no.

Take $M = \langle \{ 0, \dots, m\}, < \rangle$ and $N = \langle \{ 0, \dots, n\}, < \rangle$. Both have only one automorphism, namely Id. So their automorphism groups are isomorphic, but the structures are not.

Reconstruction problems (cont.)

In order to achieve $(\star)$, one has to assume that $\mathrm{Aut}(M)$ and $\mathrm{Aut}(M)$ are "sufficiently rich". Here is an example of such a theorem.

Theorem 1 (J. Whittaker 1963) Let X and Y be open subsets of respectively $\mathbb{R}^m$ and $\mathbb{R}^n$ (or more generally Euclidean manifolds). If $\phi : \mathrm{H}(X) \cong \mathrm{H}(Y)$, then there exists a homeomorphism τ between X and Y such that

$$\phi(g) = \tau \circ g \circ \tau^{-1} \text{ for all } g \in \mathrm{H}(X)$$

Reconstruction problems (cont.)

A pair of the form $\langle M, G \rangle$ where M is a model and $G \leq \mathrm{Aut}(M)$ is called an MG-pair. A class of MG-pairs in which all the M's are models in the the same language is called an MG-class.

Definition 2 An MG-class K is said to be faithful, if for every $\langle M, G \rangle, \langle N, H \rangle \in K$, and $\phi : G \cong H$ there is $\psi : M \cong N$ such that ψ induces ϕ.

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- Note that the property of being faithful is meaningful even if K is a singleton. Namely: Suppose that $\{\langle M, G \rangle\}$ is faithful. Then for every $\phi \in \mathrm{Aut}(G)$, there is $\psi \in \mathrm{Aut}(M)$ such that ϕ is the function

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•• Theorems which prove that a certain class is faithful are called faithfulness theorems or reconstruction theorems.

A reconstruction theorem for locally compact spaces

We shall "prove" a strengthening of Whittaker's Theorem.

A pair $\langle X, G \rangle$, where X is a regular space and $G \leq \mathrm{H}(X)$ is called a space-group pair.

We define a class K_{LC} of space-group pairs which will be proved to be faithful.

A subset $D \subseteq X$ is somewhere dense, if there is a nonempty open set $U \subseteq X$ such that $D \cap U$ is dense in U.

K_{LC} is the class of all space-group pairs $\langle X, G \rangle$ such that

(1) X is locally compact without isolated points.

(2) For every open $U \subseteq X$ and $x \in U$, $\{g(x) \mid g \in G \text{ and } g{\restriction}(X \setminus U) = \mathrm{Id}\}$ is somewhere dense.

Theorem 3 (1989) K_{LC} is faithful.

A simple example of a faithful class

Example Let A be a nonempty set. $M_A := \langle A, =^M \rangle$ denotes the structure whose universe is A, and which does not have any relations and any operations other than equality. Clearly, $\mathrm{Aut}\,(M_A) = \mathrm{Sym}\,(A)$.

A simple example of a faithful class (cont.)

Theorem 4 (Who proved this?) Let $K_{\approx} = \{\langle M_A, \mathrm{Sym}(A)\rangle \mid |A| \neq 6\}$.

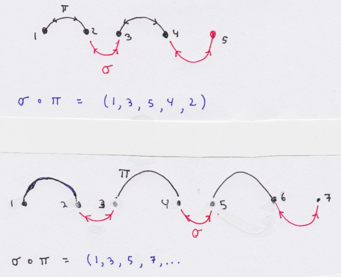
Then $K_{\approx}$ is faithful.

Proof Let $\varphi_{\mathrm{tr}}(x)$ be the following formula in the language of groups.

$$(\mathrm{or}(x) = 2) \wedge (\forall y \sim x)((x \approx y) \vee (\mathrm{or}(xy) = 2) \vee (\mathrm{or}(xy) = 3)).$$

Then for every A such that $|A| \neq 6$, and for every $g \in \mathrm{Sym}(A)$,

$\mathrm{Sym}(A) \models \varphi_{\mathrm{tr}}[g]$ iff g is a transposition.



A simple example of a faithful class (cont.)

Let $a \in A$ and g_1, g_2 be transpositions. We say that $\langle g_1, g_2 \rangle$ represent a, if the intersection of the supports of g_1 and g_2 is $\{a\}$. We then write: $\mathrm{pt}\,(g_1, g_2) = a$. Let $\varphi_{\mathrm{pt}}\,(x_1, x_2)$ be the formula

$$x_1 x_1 \neq x_2 x_1 .$$

Clearly, two transposition represent some point iff they satisfy φ_{pt}.

Next we need a formula $\varphi_{\approx}(x_1, x_2, x_3, x_4)$ which expresses the fact that $\mathrm{pt}\,(x_1, x_2) = \mathrm{pt}\,(x_3, x_4)$. Define $\varphi_{\approx}(x_1, \ldots, x_4)$ to be the following formula:

(1) For every distinct $x, y \in \{ x_1, \ldots, x_4\}$, $\varphi_{\mathrm{pt}}\,(x, y)$

and

(2) If $|\{ x_1, \ldots, x_4\}| = 3$, then $\mathrm{or}\,(\prod\{ x_1, \ldots, x_4\}) = 4$.

A simple example of a faithful class (cont.)

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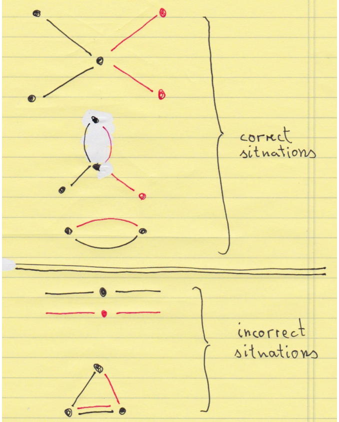


A simple example of a faithful class (cont.)

Let $\phi : \mathrm{Sym}(A) \cong \mathrm{Sym}(B)$. We define $\tau : A \to B$.

Let $a \in A$. Choose a representing pair $\langle f_1, f_2 \rangle$ such that $\mathrm{pt}(f_1, f_2) = a$. Then $\mathrm{Sym}(A) \models \varphi_{\mathrm{pt}}[f_1, f_1]$. So $\mathrm{Sym}(B) \models \varphi_{\mathrm{pt}}[\phi(f_1), \phi(f_1)]$. So $\langle \phi(f_1), \phi(f_2) \rangle$ is a representing pair. Define $\tau(a)$ to be

$$\tau(a) = \mathrm{pt}(\phi(f_1), \phi(f_1)).$$

Suppose that $\langle \phi(g_1), \phi(g_2) \rangle$ is another representing pair such that $\mathrm{pt}(g_1, g_2) = a$. Then $\mathrm{Sym}(A) \models \varphi_{\approx}[f_1, f_2, g_1, g_2]$. So $\mathrm{Sym}(B) \models \varphi_{\approx}[\phi(f_1), \phi(f_2), \phi(g_1), \phi(g_2)]$. So

$$\mathrm{pt}(\phi(f_1), \phi(f_2)) = \mathrm{pt}(\phi(g_1), \phi(g_2)).$$

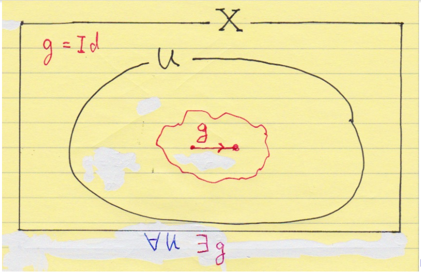
This means that τ is well defined.

We skip the proof that τ is a bijection and that ϕ is the function $h \mapsto \tau \circ h \circ \tau^{-1}$.

Locally moving groups, definition and examples

Whittaker's Reconstruction Theorem and all the reconstruction theorems that will be mentioned later in this lecture are based on a notion called a "locally moving group". We define this notion and state the theorem which serves as "Step 1" in all these theorems. This theorem is called "The reconstruction theorem for locally moving groups".

Definition 5 Let X be a regular space and $G \leq \mathrm{H}(X)$. We say that G is a locally moving (LM) group for X, if for every nonempty open subset U of X there is $g \in G \setminus \{\mathrm{Id}_B\}$ such that $g \restriction (X \setminus U) = \mathrm{Id}$.



Locally moving groups, definition and examples (cont.)

If G is a locally moving group for X, then the pair $\langle X, G \rangle$ is called a topological LM-pair.

Let H be a group. The group H is called a topologically locally moving (LM) group, if there is a topological LM-pair $\langle X, G \rangle$ such that $H \cong G$.

Locally moving groups, definition and examples (cont.)

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- Note that if $\langle X, G \rangle$ is a topological LM-pair, then X has no isolated points.

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• Note that if $\langle X, G \rangle$ is a topological LM-pair, then X has no isolated points.

•• Note that if $\langle X, G \rangle$ is a topological LM-pair and $G \leq H \leq \mathrm{H}(X)$, then $\langle X, H \rangle$ too is a topological LM-pair.

Locally moving groups, definition and examples (cont.)

Examples: The following groups are topologically locally moving.

(1) The group of auto-homeomorphisms of any nonempty open subset of any Euclidean space.

(2) The Thompson Group $=$ The group of all piecewise linear homeomorphisms g of $[0,1]$ such that every slope of g is an integral power of 2, and every break-point of g is a dyadic number.

(3) The automorphism group of the binary tree.
Let α be a limit ordinal and $\mu > 1$ be a cardinal. Let $\mathbb{T}_{\alpha,\mu}$ be the tree of all sequences with entries in μ and with length $< \alpha$. Then $\mathrm{Aut}\,(\mathbb{T}_{\alpha,\mu})$ is localy moving.

(4) The group of measure preserving automorphisms of the quotient of the Borel field of $[0,1]$ over the ideal of measure 0 sets.

Locally moving groups, definition and examples (cont)

(5) The group of bilipschitz homeomorphisms of the Urysohn space.

Locally moving groups, definition and examples (cont)

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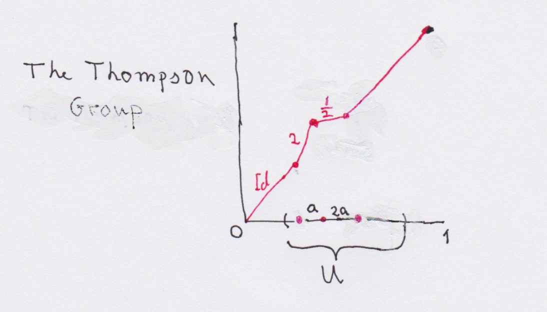
- 

The Boolean algebraic definition of an LM group

Definition 5 Let B be a complete atomless Boolean algebra and G be a subgroup of $\mathrm{Aut}(B)$. We say that G is a locally moving (LM) group for B if for every nonzero $a \in B$ there is $g \in G \setminus \{\mathrm{Id}_B\}$ such that for every $b \in B$: if $a \cdot b = 0^B$, then $g(b) = b$.

We call $\langle B, G \rangle$ an LM-pair.

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Observation 6 For every group G, the following statements are equivalent.

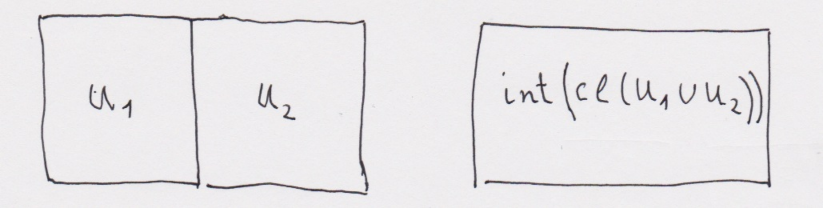
(1) There is a regular space X and a locally moving group H for X such that G is isomorphic to H.

(2) There is an atomless complete Boolean algebra B and a locally moving group H for B such that G is isomorphic to H.

A group satisfying (1) or (2) above, is called a locally moving group.

The Boolean algebraic definition of an LM group (cont.)

An open subset U of a topological space X is regular open, if $U = \mathrm{int}(\mathrm{cl}(U))$.



For a topological space X, let $\mathrm{Ro}(X)$ denote the set of regular open subsets of X. Define the following operations on $\mathrm{Ro}(X)$.

(1) $U \cdot V = U \cap V$,

(2) $U + V = \mathrm{int}(\mathrm{cl}(U \cup V))$ and

(3) $-U = \mathrm{int}(X \setminus U)$.

$\mathbf{Ro}(X) := \langle \mathrm{Ro}(X), \cdot, +, - \rangle$ is a complete Boolean algebra, and $\mathbf{Ro}(X)$ is atomless iff X has no isolated points.

The Boolean algebraic definition of an LM group (cont.)

Proof of Observation 5 Suppose that G is locally moving for X. Then G is locally moving for $\mathbf{Ro}(X)$.

Suppose that G is locally moving for B. Then G is locally moving for the space of ultrafilters of B ($=$ the Stone space of B). ∎

The Reconstruction Theorem for LM groups

Let $\langle B, G \rangle$ be an LM-pair. We define a structure based on B and G. We call this structure <u>The action structure of B and G</u> and denote it by $\mathrm{ACT}(B, G)$.

(1) $|\mathrm{ACT}(B, G)| = B \cup G$.

(2) The relations of $\mathrm{ACT}(B, G)$ are equality and $\leq^B$.

(3) The operations of $\mathrm{ACT}(B, G)$ are $\circ^G$ and the application operation, namely,

$$\mathrm{Ap}^{(B,G)}(g, a) = g(a), \quad g \in G \quad \text{and} \quad a \in B.$$

The Reconstruction Theorem ... (Cont.)

Let K_{LM} be the class of all LM-pairs. Set

$$K^{ACT}_{\mathrm{LM}} := \{\mathrm{ACT}(B, G) \mid \langle B, G \rangle \in K_{\mathrm{LM}}\}$$

and

$$K^{Gr}_{\mathrm{LM}} := \{G \mid \text{there is } B \text{ such that } \langle B, G \rangle \in K_{\mathrm{LM}}\}.$$

Theorem 7 (The Reconstruction Theorem for LM Groups) Let $\langle B, G \rangle, \langle C, H \rangle \in K_{\mathrm{LM}}$ and $\phi : G \cong H$. Then there is $\psi : B \cong C$ such that

$$\psi \cup \phi : \mathrm{ACT}(B, G) \cong \mathrm{ACT}(C, H).$$

The Reconstruction Theorem ... (Cont.)

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$$\psi \cup \phi : \mathrm{ACT}(B,G) \cong \mathrm{ACT}(C,H).$$

- Saying that $\psi \cup \phi : \mathrm{ACT}(B,G) \cong \mathrm{ACT}(C,H)$ is equivalent to saying that ϕ is the function $g \mapsto \psi \circ g \circ \psi^{-1}$. So Theorem 7 can be restated as follows.

Theorem 7* K_{LM} is faithful.

The second order extension of a structure

Definition 8 Let $\mathcal{L}$ be a f.o. language. $\mathcal{L}^{\text{II}}$ is the language obtained from $\mathcal{L}$ by adding to $\mathcal{L}$ a binary relation symbol ε^1 and a 3-place relation symbol ε^2. If M is an $\mathcal{L}$-structure, then M^{II} is the structure $(M, \mathrm{P}(|M|), \mathrm{P}(|M|^2), \varepsilon^1_M, \varepsilon^2_M)$, where

$\varepsilon^1_M = \{\langle a, P\rangle \mid a \in |M|, P \in \mathrm{P}(|M|)$ and $a \in P\}$ and

$\varepsilon^2_M = \{\langle a, b, P\rangle \mid a, b \in |M|, P \in \mathrm{P}(|M|^2)$ and $\langle a, b\rangle \in P\}$.

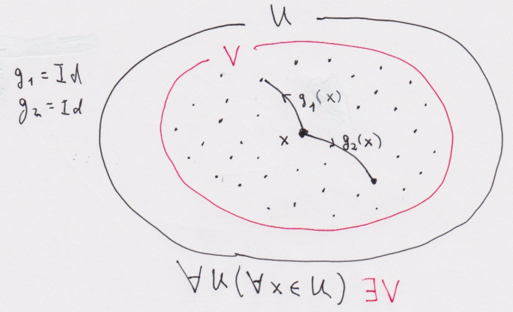
Clearly, if $\phi : M \cong N$, there there is a unique $\phi^{\mathrm{II}} : M^{\mathrm{II}} \cong^{\mathrm{II}} N$ such that $\phi^{\mathrm{II}} \supseteq \phi$.

A reconstruction theorem for locally compact spaces

We prove Theorem 3. Recall that Theorem 3 says that the class K_{LC} is faithful. K_{LC} is the class of all space-group pairs $\langle X, G \rangle$ such that

(1) X is locally compact without isolated points.

(2) For every open $U \subseteq X$ and $x \in U$,
$\{g(x) \mid g \in G \text{ and } g{\restriction}(X \setminus U) = \mathrm{Id}\}$ is somewhere dense.



The faithfulness of K_{LC}

We describe informally the general plan of the proof, omitting many of the details. Let $\langle X, G \rangle \in K_{\mathrm{LC}}$. We represent the points of X by certain ultrafilters in $\mathrm{Ro}(X)$. These ultrafilters are called good ultrafilters. Note that $\mathrm{Ult}(\mathrm{Ro}(X)) \subseteq (\mathrm{ACT}(\mathrm{Ro}(X), G)^{\mathrm{II}}$.

The following two facts play a role in the proof:

(1) If p is a good ultrafiler, then there is $x \in X$ such that $\bigcap_{U \in p} \mathrm{cl}^X(U) = \{x\}$.

(2) For every $x \in X$ there is a good ultrafilter p such that $\bigcap_{U \in p} \mathrm{cl}^X(U) = \{x\}$.

For a good ultrafilter p we denote by a_p^X that element x such that $\bigcap_{U \in p} \mathrm{cl}^X(U) = \{x\}$.

The faithfulness of K_{LC} (cont.)

The third important fact is

(3) The property of being good is expressible by a first order formula $\varphi_{\mathrm{good}}(x)$ in $(\mathrm{ACT}(\mathrm{Ro}(X), G)^{\mathrm{II}}$.

So being good is preserved under isomorphisms between $(\mathrm{ACT}(\mathrm{Ro}(X), G)^{\mathrm{II}}$ and $(\mathrm{ACT}(\mathrm{Ro}(Y), H)^{\mathrm{II}}$.

Consider the following property $\mathrm{Eq}(p, q)$ of a pair of good ultrafilters p and q:

$$\mathrm{Eq}(p, q) \quad \Leftrightarrow \quad a_p^X = a_q^X.$$

(4) The property $\mathrm{Eq}(p, q)$ is also expressible by a first order formula $\varphi_{\mathrm{Eq}}(x, y)$ in $(\mathrm{ACT}(\mathrm{Ro}(X), G)^{\mathrm{II}}$.

Hence $\mathrm{Eq}(p, q)$ is preserved under isomorphisms between $(\mathrm{ACT}(\mathrm{Ro}(X), G)^{\mathrm{II}}$ and $(\mathrm{ACT}(\mathrm{Ro}(Y), H)^{\mathrm{II}}$.

The faithfulness of K_{LC} (cont.)

Let $\langle X, G\rangle, \langle Y, H\rangle \in K_{\mathrm{LC}}$ and $\phi : G \cong H$. By Theorem 7, there is $\hat{\psi} : \mathrm{Ro}(X) \to \mathrm{Ro}(Y)$ such that $\hat{\psi} \cup \phi : \mathrm{ACT}(\mathrm{Ro}(X), G) \cong \mathrm{ACT}(\mathrm{Ro}(Y), H)$. Let $\bar{\psi} = \hat{\psi} \cup \phi$ and ψ be the extension of $\bar{\psi}$ to $(\mathrm{ACT}(\mathrm{Ro}(X), G))^{\mathrm{II}}$.

We now define a bijection τ between X and Y. Let $x \in X$. Choose a good ultrafilter p such that $a_p^X = x$. Define $\tau(x) = a_{\psi(p)}^Y$. Since $\mathrm{Eq}(p, q)$ is preserved under isomorphisms, τ is well-defined. One has to show that τ is a homeomorphism between X and Y. For this one needs the following fact.

(5) For a good ultrafilter p and $U \in \mathrm{Ro}(X)$ the property

$$a_p^X \in U$$

is expressible by a first order formula $\varphi_{\in}(x, y)$ in $(\mathrm{ACT}(\mathrm{Ro}(X), G)^{\mathrm{II}}$.

The faithfulness of K_{LC} (cont.)

We fill in the details, and give a more formal argument.

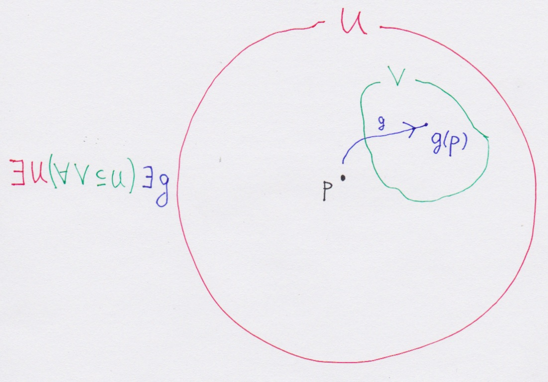
Claim 1 For $p \in \mathrm{Ult}(\mathrm{Ro}(X))$ define $A_p^X = \bigcap_{U \in p} \mathrm{cl}^X(U)$. Then A_p^X is either a singleton or the empty set.

Proof Let $x, y \in X$ be distinct. There are regular open sets $U \in \mathrm{Nbr}(x)$ and $V \in \mathrm{Nbr}(y)$ such that $U \cap V = \emptyset$. Either $U \notin p$ or $V \notin p$. If the former happens then $x \notin A_p^X$, and if the latter happens then $y \notin A_p^X$. So $|A_p^X| \leq 1$. ∎

The faithfulness of K_{LC} (cont.)

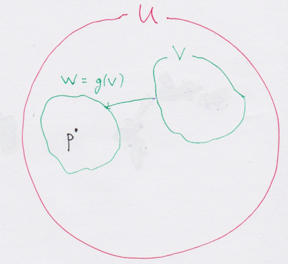
We say that $p \in \mathrm{Ult}(\mathrm{Ro}(X))$ is a good ultrafilter if the following holds:

There is $U \in \mathrm{Ro}(X) \setminus \{\emptyset\}$ such that for every $V \in (\mathrm{Ro}(X){\restriction}U) \setminus \{\emptyset\}$ there is $g \in G$ such that $V \in g(p)$.



The faithfulness of K_{LC} (cont.)

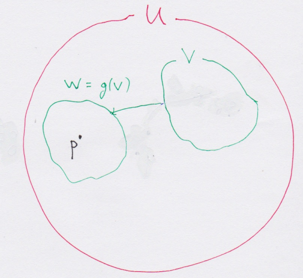
Claim 2 If p is good, then there is $x \in X$ such that $A_p^X = \{x\}$. **Proof** Let p be good and U be as assured by the goodness of p. There is $V \in (\mathrm{Ro}(X) \upharpoonright U) \setminus \{\emptyset\}$ such that $\mathrm{cl}(V)$ is compact. Let $g \in G$ be such that $W := g(V) \in p$. So $\mathrm{cl}(W)$ is compact.



For every $S \in p$, $\mathrm{cl}(S \cap W) \subseteq \mathrm{cl}(S)$. So

$$A := \bigcap_{S \in p} \mathrm{cl}(S \cap W) \subseteq \bigcap_{S \in p} \mathrm{cl}(S) = A_p^X .$$

The faithfulness of K_{LC} (cont.)



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Now, $\{\mathrm{cl}(S \cap W) \mid S \in p\}$ is a family of compact sets with the finite intersection property. This is so, since $\{S \cap W \mid S \in p\} \subseteq p$. So $A \neq \emptyset$ and hence $A_p^X \neq \emptyset$. ■

The faithfulness of K_{LC} (cont.)

Suppose that p is a good ultrafilter and $A_p^X = \{a\}$. Then a is denoted by a_p^X

If p is good and $g \in G$ then $g(p)$ is good. This is so, since $g \in \mathrm{H}(X)$.

Also, because of the same reason, $g(a_p^X) = a_{g(p)}^X$.

Let $\mathrm{Good}(X, G)$ denote the set of good ultafilters of $\langle X, G \rangle$.

Claim 3 For every $x \in X$ there is $p \in \mathrm{Good}(X, G)$ such that $a_p^X = x$.

Proof We show the following facts:

(F1) For every $x \in X$ there is $p \in \mathrm{Ult}(\mathrm{Ro}(X))$ such that $A_p^X = \{x\}$.

(F2) Suppose that $p \in \mathrm{Ult}(\mathrm{Ro}(X))$ and $A_p^X = \{x\}$. Then $p \in \mathrm{Good}(X, G)$.

Clearly, Facts (F1) and (F2) imply Claim 3.

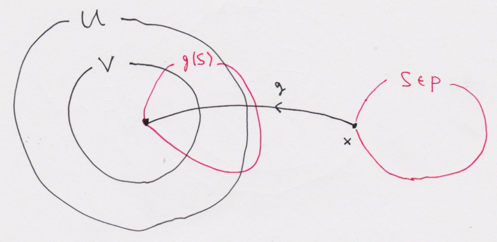
The faithfulness of K_{LC} (cont.)

Let $x \in X$ and define $r := \mathrm{Nbr}^X(x) \cap \mathrm{Ro}(X)$. Then $r \subseteq \mathrm{Ro}(X)$, and r has the finite intersection property. So there is $p \in \mathrm{Ult}(\mathrm{Ro}(X))$ such that $p \supseteq r$. Obviously, for every $U \in p$, $x \in \mathrm{cl}(U)$. So $A^X_p = \{x\}$. We have proved (F1).

The faithfulness of K_{LC} (cont.)

(F2) Suppose that $p \in \mathrm{Ult}(\mathrm{Ro}(X))$ and $A_p^X = \{x\}$. Then $p \in \mathrm{Good}(X, G)$.

So let $p \in \mathrm{Ult}(\mathrm{Ro}(X))$ be such that $A_p^X = \{x\}$. Let U be such that $\mathrm{Orb}(x, G)$ is dense in U. We show that U is as required in the definition of goodness. Let $V \in (\mathrm{Ro}(X) \restriction U) \setminus \{\emptyset\}$. So there is $g \in G$ such that $g(x) \in V$. It is trivial to verify that $\{V\} \cup g(p)$ has the finite intersection property. Since p is an ultrafilter, $g^{-1}[V]$ must belong to p.



This proves (F2), and hence Claim 3 is proved. ■

The faithfulness of K_{LC} (cont.)

We shall also use the following trivial observation.

(F3) If $p \in \mathrm{Ult}(\mathrm{Ro}(X))$ and $A_p^X = \{x\}$, then $\mathrm{Nbr}(x) \cap \mathrm{Ro}(X) \subseteq p$.

Next we wish to show that:

$(*)$ If $\eta : \mathrm{ACT}(\mathrm{Ro}(X), G) \cong \mathrm{ACT}(\mathrm{Ro}(Y), H)$ and $p \in \mathrm{Ult}(\mathrm{Ro}(X))$ is good with respect to X and G, then $\eta(p)$ is good with respect to Y and H.

In fact, we shall show that

$(**)$: There is a first order formula $\varphi_{\mathrm{Good}}(x)$ in the language of $(\mathrm{ACT}(\mathrm{Ro}(X), G))^{\mathrm{II}}$, which is equivalent to the fact that x is good.

The faithfulness of K_{LC} (cont.)

Recall that

$$\mathrm{Act}(\mathrm{Ro}(X), G) = (\mathrm{Ro}(X), G; +^{\mathrm{Ro}(X)}, \cdot^{\mathrm{Ro}(X)}, \circ^{G}, \mathrm{Ap}^{G,\mathrm{Ro}(X)}).$$

Here is the formula in $\mathcal{L}((\mathrm{ACT}(\mathrm{Ro}(X), G))^{\mathrm{II}})$ which expresses the fact that p is good.

$(p$ is an ultrafilter$)$ $\wedge$

$(\exists U \in \mathrm{Ro}(X))$

$$\Bigg((U \neq 0) \ \wedge \ (\forall V \subseteq U)\bigg((V \neq 0) \to (\exists g \in G)\Big(V \in \mathrm{Ap}(g, p)\Big)\bigg)\Bigg).$$

The faithfulness of K_{LC} (cont.)

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$$\Big((U \neq 0) \ \wedge \ (\forall V \subseteq U)\Big((V \neq 0) \to (\exists g \in G)\Big(V \in \mathrm{Ap}(g, p)\Big)\Big)\Big).$$

Claim 4 Let $\eta : \mathrm{ACT}(\mathrm{Ro}(X), G) \cong \mathrm{ACT}(\mathrm{Ro}(Y), H)$. and let p be a good ultrafilter in $\mathrm{ACT}(\mathrm{Ro}(X), G)$. Then $\eta(p)$ is a good ultrafilter in $\mathrm{ACT}(\mathrm{Ro}(Y), H)$,

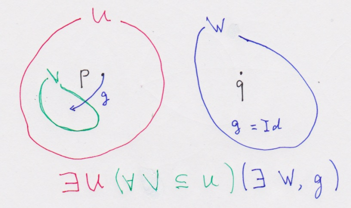
The faithfulness of K_{LC} (cont.)

Our next goal is to express the fact that $a_p^X = a_q^X$ by a first order formula in $(\mathrm{ACT}(\mathrm{Ro}(X), G))^{\mathrm{II}}$.

Let $g \in G$ and $V \in \mathrm{Ro}(X)$. Set $g \restriction V := g \restriction \{W \in \mathrm{Ro}(X) \mid W \subseteq V\}$.

Consider the following property $\mathrm{Diff}(p, q)$ of p and q:

> There is $U \in p$ such that for every $V \in (\mathrm{Ro}(X) \restriction U) \setminus \{\emptyset\}$ there are $W \in q$ and $g \in G$ such that $g \restriction W = \mathrm{Id}$ and $V \in g(p)$.



Note that there is a formula $\varphi_{\mathrm{Diff}}(x, y)$ in $\mathcal{L}((\mathrm{ACT}(\mathrm{Ro}(X), G))^{\mathrm{II}})$ which expresses the property $\mathrm{Diff}(x, y)$.

The faithfulness of K_{LC} (cont.)

Claim 5 Let $p, q \in \mathrm{Good}(X, G)$. Then the following are equivalent:

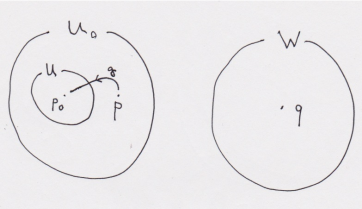
(1) $a_p^X \neq a_q^X$.

(2) $\mathrm{Diff}(p, q)$ holds.

Proof Let p, q be good ultrafilters, and suppose that $x := a_p^X \neq a_q^X := y$.

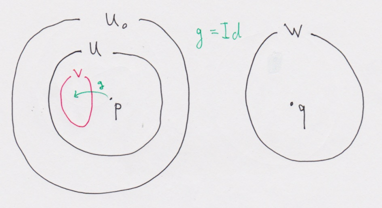
Let U_0 and W be disjoint regular open neighborhoods of respectively x and y.

There is $U \in \mathrm{Ro}^+(X) \restriction U_0$ be such that $G\left|\underline{U_0}\right|(x)$ is dense in U. We may assume that $x \in U$.



The faithfulness of K_{LC} (cont.)

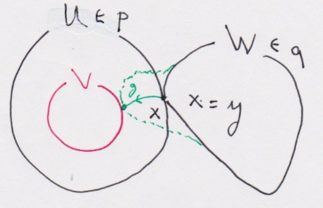
Let $V \subseteq U$ be a nonempty regular open set.



There is $g \in G\lfloor U_0 \rfloor$ such that $g(x) \in V$. Then $V \in p$. Also $g \restriction W = \mathrm{Id}$. So $\mathrm{Diff}(p, q)$ holds.

The faithfulness of K_{LC} (cont.)

Now assume that $x = y$, and we show that $\mathrm{Diff}(p, q)$ does not hold.



Suppose that $U \in p$ and $W \in q$. Then $x \in \mathrm{cl}(W)$. Let V be a nonempty regular open set such that $V \subseteq U$ and $x \notin \mathrm{cl}(V)$. For every $g \in G$: if $V \in g(p)$, then $g(x) \neq x$. It follows that for every such g, $g \restriction W \neq \mathrm{Id}$. ■

Let $\mathrm{Eq}(p, q) = \neg\mathrm{Diff}(p, q)$. Then there is a formula $\varphi_{\mathrm{Eq}}(u, v)$ in $\mathcal{L}((\mathrm{ACT}(\mathrm{Ro}(X), G))^{\mathrm{II}})$ which expresses the property $\mathrm{Eq}(u, v)$. Hence

$$a_p^X = a_q^X \quad \text{iff} \quad (\mathrm{ACT}(\mathrm{Ro}(X), G))^{\mathrm{II}} \models \varphi_{\mathrm{Eq}}[p, q].$$

The faithfulness of K_{LC} (cont.)

We are ready to define τ and to show that it is a bijection between X and Y. But we need an additional ingredient in order to prove that τ is a homeomorphism between X and Y. We deal with this ingredient first.

The faithfulness of K_{LC} (cont.)

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Claim 6 Let $U \in \mathrm{Ro}(X)$ and $p \in \mathrm{Good}(X, G)$. Then the following are equivalent:

(1) $a_p^X \in U$.

(2) $(\forall q \in \mathrm{Good}(X, G))(\varphi_{\mathrm{Eq}}(p, q) \to (U \in q))$.

The faithfulness of K_{LC} (cont.)

Claim 6 Let $U \in \mathrm{Ro}(X)$ and $p \in \mathrm{Good}(X, G)$. Then the following are equivalent:

(1) $a_p^X \in U$.

(2) $(\forall q \in \mathrm{Good}(X, G))(\varphi_{\mathrm{Eq}}(p, q) \to (U \in q))$.

Proof Suppose that $a_p^X \in U$ and $\varphi_{\mathrm{Eq}}(p, q)$ holds. Then $a_q^X = a_p^X$. Since $U \in \mathrm{Nbr}(a_q^X) \cap \mathrm{Ro}(X)$, it follows that $U \in q$.

Suppose that $a_p^X \notin U$. Then $r := (\mathrm{Nbr}(a_p^X) \cap \mathrm{Ro}(X)) \cup \{\mathrm{int}(X \setminus U)\}$ has the finite intersection property. Let $q \in \mathrm{Ult}(\mathrm{Ro}(X))$ be such that $q \supseteq r$. Then $A_q^X = \{a_p^X\}$. So q is good. Also $\varphi_{\mathrm{Eq}}(p, q)$ holds. But $U \notin q$. So (2) does not hold. ∎

Let $\varphi_{\in}(p, U)$ denote the formula

$$(\forall q \in \mathrm{Good}(X, G))(\varphi_{\mathrm{Eq}}(p, q) \to (U \in q)).$$

The faithfulness of K_{LC} (cont.)

Summary There are first order formulas $\varphi_{\mathrm{Good}}(p)$, $\varphi_{\mathrm{Eq}}(p,q)$ and $\varphi_{\in}(p,U)$ in $\mathcal{L}((\mathrm{ACT}(\mathrm{Ro}(X),G))^{\mathrm{II}})$ such that for every $p,q \in \mathcal{P}(\mathrm{Ro}(X))$ and $U \in \mathrm{Ro}(X)$

(1) $(\mathrm{ACT}(\mathrm{Ro}(X),G))^{\mathrm{II}} \models \varphi_{\mathrm{Good}}[p]$ <u>iff</u> p is a good ultrafilter.

(2) Suppose that p,q are good ultrafilters. Then

$$(\mathrm{ACT}(\mathrm{Ro}(X),G))^{\mathrm{II}} \models \varphi_{\mathrm{Eq}}[p,q] \quad \underline{\text{iff}} \quad a_p^X = a_q^Y.$$

(3) Suppose that p is a good ultrafilter and $U \in \mathrm{Ro}(X)$. Then

$$(\mathrm{ACT}(\mathrm{Ro}(X),G))^{\mathrm{II}} \models \varphi_{\in}[p,U] \quad \underline{\text{iff}} \quad a_p^X \in U.$$

It follows that for every $\eta^{\mathrm{II}} : (\mathrm{ACT}(\mathrm{Ro}(X),G))^{\mathrm{II}} \cong (\mathrm{ACT}(\mathrm{Ro}(Y),H))^{\mathrm{II}}$:

(1) p is good iff $\eta^{\mathrm{II}}(p)$ is good,

(2) $a_p^X = a_q^X$ iff $a_{\eta^{\mathrm{II}}(p)}^Y = a_{\eta^{\mathrm{II}}(q)}^Y$, and

(3) $a_p^X \in U$ iff $a_{\eta^{\mathrm{II}}(p)}^Y \in \eta^{\mathrm{II}}(U)$.

The faithfulness of K_{LC} (cont.)

We now define τ. Suppose that $\varphi : G \cong H$ is given. We concluded that there is $\psi : \mathrm{Ro}(X) \cong \mathrm{Ro}(X)$ such that $\eta := \varphi \cup \psi$ is an isomorphism between $\mathrm{ACT}(\mathrm{Ro}(X), G)$ and $\mathrm{ACT}(\mathrm{Ro}(Y), H)$ Clearly is a unique $\eta^{\mathrm{II}} : (\mathrm{ACT}(\mathrm{Ro}(X), G)^{\mathrm{II}} \cong (\mathrm{ACT}(\mathrm{Ro}(Y), H)^{\mathrm{II}}$ such that $\eta^{\mathrm{II}} \supseteq \eta$.

The faithfulness of K_{LC} (cont.)

We now define τ. Suppose that $\varphi : G \cong H$ is given. We concluded that there is $\psi : \mathrm{Ro}(X) \cong \mathrm{Ro}(X)$ such that $\eta := \varphi \cup \psi$ is an isomorphism between $\mathrm{ACT}(\mathrm{Ro}(X), G)$ and $\mathrm{ACT}(\mathrm{Ro}(Y), H)$ Clearly is a unique $\eta^{\mathrm{II}} : (\mathrm{ACT}(\mathrm{Ro}(X), G)^{\mathrm{II}} \cong (\mathrm{ACT}(\mathrm{Ro}(Y), H)^{\mathrm{II}}$ such that $\eta^{\mathrm{II}} \supseteq \eta$.

- Let $x \in X$. By Claim 3, there is $p \in \mathrm{Good}(X, G)$ such that $a^X_p = x$. Then $\eta^{\mathrm{II}}(p) \in \mathrm{Good}(Y, H)$. Define $\tau(x) = a^Y_{\eta^{\mathrm{II}}(p)}$. The definition of τ is valid, since if q is another good ultrafilter such that $a^X_q = x$, then $a^Y_{\eta^{\mathrm{II}}(q)} = a^Y_{\eta^{\mathrm{II}}(p)}$. It is to verify that τ is a bijection between X and Y.

The faithfulness of K_{LC} (cont.)

We show that τ is a homeomorphism between X and Y. Indeed we show that for every $U \in \mathrm{Ro}(X)$,

$$(*) \qquad \tau[U] = \eta^{\mathrm{II}}(U).$$

Let $x \in X$ and let $p \in \mathrm{Good}(X, G)$ be such that $a_p^X = x$. Then

$x \in U$ iff

$a_p^X \in U$ iff

$(\mathrm{ACT}(\mathrm{Ro}(X), G))^{\mathrm{II}} \models \varphi_{\in}[p, U]$ iff

$(\mathrm{ACT}(\mathrm{Ro}(X), G))^{\mathrm{II}} \models \varphi_{\in}[\eta^{\mathrm{II}}(p), \eta^{\mathrm{II}}(U)]$ iff

$a^Y_{\eta^{\mathrm{II}}(p)} \in \eta^{\mathrm{II}}(U)$ iff

$\tau(x) \in \eta^{\mathrm{II}}(U)$.

We have proved $(*)$.

So $\tau[U]$ is open for every regular open U. The same is true for τ^{-1}. So τ is a homeomorphism.

Consequences of the Reconstruction Theorem

Theorem 9 Let L be the class of all unary trees M which satisfy the following conditions:

(1) M is complete.

(2) For every $s \in |M|$, $|\mathrm{Suc}(s)| \neq 1$. (I.e. s never has a unique successor).

(3) For every $s \in |M|$: either for every $u, v \in \mathrm{Suc}(s)$, $u \sim^M v$; or for every distinct $u, v \in \mathrm{Suc}(s)$, $u \not\sim^M v$.

(4) For every $s \in |M|$ and $t > s$: if $|\mathrm{Orb}(t; s)| \leq 2$, then $t \in \mathrm{Suc}(s) \setminus \mathrm{Max}(M)$.

Then L is faithful.

Consequences of the Reconstruction Theorem (cont.)

Let X be a nonempty open subset of a Banach space E. Denote by $\mathrm{Diff}^k(X)$ the group of all auto-homeomorphisms g of X such that g and g^{-1} are k times continuously differentiable.

Theorem 10 (Y. Yomdin, M. Rubin) Let E, F be Banach spaces whose norms are k times continuously differntiable away from 0. Let $X \subseteq E$ and $Y \subseteq F$ be open and nonempty, and $\phi : \mathrm{Diff}^k(X) \cong \mathrm{Diff}^k(Y)$. Then there is $\tau : X \cong Y$ such that τ and τ^{-1} are k times continuously differentiable and such that ϕ is conjugation by τ.

An open question

A topological vector space is a vector space E over $\mathbb{R}$ equipped with a topology σ such that the operations of E:

$$+^E : E \times E \to E, \qquad \cdot^E : \mathbb{R} \times E \to E \qquad \text{and} \qquad -^E : E \to E$$

are continuous with respect to σ and $\sigma^{\mathbb{R}}$.

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Conjecture 1 Let

$K_{\mathrm{LCVX}} = \{\langle U, \mathrm{H}(U) \rangle \mid U$ is an open subset of a locally convex space$\}$.

Then K_{LCVX} faithful.

An open question (Cont.)

A special case of Conjecture 1 which is also open. Prove that the class $K_{\mathrm{RPWR}} = \{\langle \mathbb{R}^{\lambda}, \mathrm{H}(\mathbb{R}^{\lambda}) \rangle \mid \lambda \text{ is an infinite cardinal}\}$ is faithful.

An open question (Cont.)

A special case of Conjecture 1 which is also open. Prove that the class $K_{\mathrm{RPWR}} = \{\langle \mathbb{R}^{\lambda}, \mathrm{H}(\mathbb{R}^{\lambda}) \rangle \mid \lambda \text{ is an infinite cardinal}\}$ is faithful.

- There are many types of locally convex spaces: Normed spaces, weak topologies on normed spaces, $\mathrm{C_p}(X)$.

Let E_λ be the vector space over $\mathbb{R}$ with Hamel base of cardinality λ. Let $\mathcal{B}_\lambda$ be the set of all convex subsets U of E_λ such that every point of U is internal point of U. And let σ_λ be the topology on E_λ whose open base is $\mathcal{B}_\lambda$. Then $\langle E_\lambda, \sigma_\lambda \rangle$ is locally convex.

Other faithfulness theorems

After the paper of Whittaker there were more faithfulness theorems.

R. McCoy 1972,

M. Rubin 1979,

W. Ling 1980,

M. Rubin 1989,

M. Rubin 1989,

K. Kawamura 1995,

M. Brin 1996,

M. Rubin 1996,

T. Rybicki 1996,

A. Banyaga 1997,

A. Leiderman and R. Rubin 1999,

M. Rubin and Y. Yomdin 2000,

J. Borzellino and V. Brunsden 2000,

T. Rybicki 2002,

Y. Maissel 2008,

V. P. Fonf and M. Rubin submitted 2013.

Other faithfulness theorems (Cont.)

Theorems on orbit equivalence of full groups of transformations are also reconstruction theorems. Such theorems were proved by

H.A. Dye, 1959 or 1963,

T. Giordano, H. Matui, I. Putnam, C. Skau 1999 and

K. Medynets 2010.

Solved special cases of Conjecture 1

Theorem 11 Let K_{NRMD} be the class of all pairs $\langle U, \mathrm{H}(U) \rangle$ such that U is an open subset of a normed space. Then K_{NRMD} is faithful.

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Let E be a topological vector space. E is bounded-on-lines, if it has a nonempty open set V such that V intersects every straight line ℓ of E in a bounded subset of ℓ.

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Theorem 12 (A. Leiderman, M. Rubin) Let K_{BOL} be the class of all pairs $\langle U, \mathrm{H}(U) \rangle$ such that U is an open subset of a normal locally convex space which is bounded-on-lines. Then K_{BOL} is faithful.

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There are non-metrizable normal locally convex spaces which are bounded-on-lines. And there are also bounded-on-lines locally convex spaces which are not normal.

Solved special cases of Conjecture 1 (cont.)

Theorem 13 (V. P. Fonf, M. Rubin) Let K_{MTRZ} be the class of all pairs $\langle U, \mathrm{H}(U) \rangle$ such that U is an open subset of a metrizable locally convex space. Then K_{MTRZ} is faithful.